Math 250 3.1 Introducing the Derivative (Find the derivative using limits!)

Objectives

1) Connections:
   a. (From 2.1) Average velocity was slope of secant line to position function and instantaneous velocity was slope of tangent line to position function.
   b. Generalize position to any function $f(x)$, and velocity to rate of change.
   c. Average rate of change = slope of secant line to $f(x)$
   d. Instantaneous rate of change = slope of tangent line to $f(x)$ = limit of the average rate of change
2) Use three different notations for slope of secant line, or average rate of change.
3) Use three different notations for slope of tangent line, or instantaneous rate of change
4) Find the slope of the tangent line to a curve at a specific point using limits
5) Write the equation of a tangent line.
6) Find the slope of the tangent line to a curve for a general point $(x, f(x))$ using limits.
7) Understand that evaluating the derivative function at a given value gives the slope of the tangent line at that x-coordinate
8) Memorize the definition of derivative (using limits).
9) IMPORTANT: If instructions say "definition of derivative" or "limit process", you MUST use limits, not shortcuts. [Caution to students who've had some calculus before: Writing just the answer will earn *no* credit.]

Big Picture: Before calculus and limits, we ask "how much is the graph increasing *between* these two points?" and find the *average* rate of change over an interval. Now we ask "how much is the graph increasing *at this point*?" and find the *instantaneous* rate of change of a function at any single point $(a, f(a))$.

First Goal: Find the slope of the tangent line to $f(x)$ at $x = a$

Slope of Tangent Line, at $x = a$, Option 1

Choose $(x_1, y_1) = (a, f(a))$ and $(x_x, y_2) = (x, f(x))$ in the slope formula to get:

Average rate of change $= m_{SEC} = \dfrac{f(x) - f(a)}{x - a}$

Instantaneous rate of change$= m_{TAN} = f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

Slope of Tangent Line, at $x = a$, Option 2

Choose $(x_1, y_1) = (a, f(a))$ and $(x_x, y_2) = (a + h, f(a + h))$ in the slope formula to get:

Average rate of change $= m_{SEC} = \dfrac{f(a+h) - f(a)}{h}$

Instantaneous rate of change $m_{TAN} = f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

Slope of Tangent Line, at $x = a$, Option 3

Choose $(x_1, y_1) = (a, f(a))$ and $(x_x, y_2) = (a + \Delta x, f(a + \Delta x))$ in the slope formula to get:

Average rate of change $= m_{SEC} = \dfrac{f(a+\Delta x) - f(a)}{\Delta x}$

Instantaneous rate of change $m_{TAN} = f'(a) = \lim\limits_{\Delta x \to 0} \dfrac{f(a+\Delta x) - f(a)}{\Delta x}$

Second Goal: Find the slope of the tangent line to $f(x)$ for any value of x -- find a function which gives this slope when we evaluate it.

Definition of Derivative, Option 1, evaluated at $x = a$

Instantaneous rate of change= $m_{TAN} = f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

Note: We can't use this option for general *x*, because we get $\dfrac{0}{0}$, the indeterminate form.

Definition of Derivative, Option 2, for any value of x

Instantaneous rate of change $m_{TAN} = f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Definition of Derivative, Option 3, for any value of x

$m_{TAN}(x) = f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

Vocabulary

The resulting function is called the derivative or first derivative, $f'(x)$, pronounced "f-prime-of-x".
Finding the derivative is called differentiating or the process of differentiation.
Differentiate of take the derivative of the function $f(x)$ to find the derivative $f'(x)$.
[CAUTION: Do not call this process "to derive", "deriving", or "derivation", which means which means the algebraic justification of a formula!]

Notation

If $y = f(x)$, the derivative can be written using many different notations:

$$f'(x) = \frac{dy}{dx} = y'(x) = \frac{d}{dx}[f(x)] = \frac{d}{dx}[y(x)] = D_x[y] = D_x[f(x)]$$

In general, the notation $\dfrac{d}{dx}$ is the operator, meaning the notation that says "take the derivative".

If we refer to the derivative evaluated at the point $x = a$, we sometimes substitute *a* for *x*, and we sometimes need a vertical bar

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = y'(a) = \left.\frac{d}{dx}[f(x)]\right|_{x=a} = \left.\frac{d}{dx}[y(x)]\right|_{x=a} = D_x[y]_{x=a} = D_x[f(x)]_{x=a}$$

Examples and Practice:

1) Using the function $f(x) = -16x^2 + 96x$,
   a. Find the slope of the tangent line at $\left(2, f(2)\right)$
   b. Write the equation of the tangent line.

2) Using the function $f(x) = \sqrt{x-3}$ and the definition of the derivative,
   a. Find $f'(x)$.
   b. Use $f'(x)$ to find the slope of the tangent line to $f(x) = \sqrt{x-3}$ when $x = 7$
   c. Write the equation of the tangent line.

3) Using the function $g(t) = \dfrac{1}{t-3}$,
   a. Find $g'(t)$ using the definition of the derivative.
   b. Find the slope of the tangent line at $\left(2, g(2)\right)$
   c. Write the equation of the tangent line.

M250

① $f(x) = -16x^2 + 96x$

a) Find slope of tangent line at $(2, f(2))$.
b) Write equation of tangent line.

$f(2) = -16(2)^2 + 96(2) = 128$

option 1: $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ where $a = 2$ $f'(a) = m_{TAN}$

$f'(2) = \lim_{x \to 2} \frac{(-16x^2 + 96x) - 128}{x - 2}$ substitute

$= \lim_{x \to 2} \frac{-16(x^2 - 6x + 8)}{(x - 2)}$ factor GCF -16

$= \lim_{x \to 2} \frac{-16(x-4)\cancel{(x-2)}}{\cancel{(x-2)}}$ factor trinomial

$= \lim_{x \to 2} -16(x-4)$ cancel

$= -16(2-4)$ subst, take limit

$= \boxed{32}$

option 2: $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ where $a = 2$ $f'(a) = m_{TAN}$

$= \lim_{h \to 0} \frac{-16(2+h)^2 + 96(2+h) - 128}{h}$ substitute

$= \lim_{h \to 0} \frac{-16(4 + 4h + h^2) + 192 + 96h - 128}{h}$ FOIL, dist

$= \lim_{h \to 0} \frac{\cancel{-64} - 64h - 16h^2 + \cancel{64} + 96h}{h}$ dist, combine

$= \lim_{h \to 0} \frac{-16h^2 + 32h}{h}$ combine

$= \lim_{h \to 0} \frac{\cancel{h}(-16h + 32)}{\cancel{h}}$ factor, cancel

$= \lim_{h \to 0} (-16h + 32)$

$= -16(0) + 32$

$= \boxed{32}$

option 3: Same as option 2 with $h = \Delta x$.

M250

b) Equation of tangent line

option 1: point-slope formula

$y - y_1 = m(x - x_1)$ $\quad \begin{cases} m = 32 & \text{from part a} \\ x_1 = 2 & \text{given} \\ y_1 = f(2) = 128 \end{cases}$

$y - 128 = 32(x-2)$ subst

$y = 32x - 64 + 128$ simplify

$\boxed{y = 32x + 64}$

option 2: slope-intercept form

$y = mx + b$ $\quad \begin{cases} m = 32 & \text{from part a} \\ x = 2 & \text{given} \\ y = f(2) = 128 \end{cases}$

$128 = 32(2) + b$

$128 = 64 + b$ solve for b.

$64 = b$

$\boxed{y = 32x + 64}$ rewrite with m & b only

② $f(x) = \sqrt{x-3}$

a) Find $f'(x)$ using definition of derivative.

b) Use $f'(x)$ to find m_{TAN} when $x = 7$.

c) Write equation of tangent line

a) Cannot use option 1 to find the function $f'(x)$

option 2: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \dfrac{\sqrt{x+h-3} - \sqrt{x-3}}{h}$ replace x by x+h

$= \lim_{h \to 0} \dfrac{(\sqrt{x+h-3} - \sqrt{x-3})}{h} \dfrac{(\sqrt{x+h-3} + \sqrt{x-3})}{(\sqrt{x+h-3} + \sqrt{x-3})}$ mult by conjugate

$= \lim_{h \to 0} \dfrac{(x+h-3) - (x-3)}{h(\sqrt{x+h-3} + \sqrt{x-3})}$ diff of squares in numerator

$= \lim_{h \to 0} \dfrac{x+h-3-x+3}{h(\sqrt{x+h-3} + \sqrt{x-3})}$ dist negative

$= \lim_{h \to 0} \dfrac{\cancel{h}}{\cancel{h}(\sqrt{x+h-3} + \sqrt{x-3})}$ combine & cancel

M250

(2) continued

a) cont.

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h-3} + \sqrt{x-3}}$$

$$= \frac{1}{\sqrt{x+0-3} + \sqrt{x-3}}$$ take limit by subst $h=0$

$$= \frac{1}{\sqrt{x-3} + \sqrt{x-3}}$$ simplify

$$= \boxed{\frac{1}{2\sqrt{x-3}}}$$ combine like radicals

b) Find $m_{TAN} = f'(7)$

$f'(x) = \frac{1}{2\sqrt{x-3}}$ from part a)

$f'(7) = \frac{1}{2\sqrt{7-3}}$ evaluate

$$= \frac{1}{2\sqrt{4}}$$

$$= \frac{1}{2 \cdot 2}$$

$$= \boxed{\frac{1}{4}}$$

c) Equation of tangent

$x = 7 \Rightarrow y = f(7) = \sqrt{7-3} = \sqrt{4} = 2$

ordered pair $(7,2)$

slope $\frac{1}{4}$ from b)

<u>option 1</u> point-slope formula

$$y - 2 = \frac{1}{4}(x-7)$$

$$y = \frac{1}{4}x - \frac{7}{4} + 2$$

$$\boxed{y = \frac{1}{4}x + \frac{1}{4}}$$

<u>option 2</u>: slope-intercept

$2 = \frac{1}{4}(7) + b \Rightarrow b = \frac{1}{4}$

$$\boxed{y = \frac{1}{4}x + \frac{1}{4}}$$

M250

③ $g(t) = \dfrac{1}{t-3}$

a) Find $g'(t)$ — from definition

b) Find slope of tangent at $(2, g(2))$

c) Write the equation of the tangent line at $(2, g(2))$

a) option 2: $g'(t) = \lim_{h\to 0} \dfrac{g(t+h)-g(t)}{h}$

$= \lim_{h\to 0} \dfrac{\frac{1}{t+h-3} - \frac{1}{t-3}}{h}$ replace t by t+h

$= \lim_{h\to 0} \dfrac{t-3-(t+h-3)}{(t-3)(t+h-3)\cdot h}$ simplify complex fraction

$= \lim_{h\to 0} \dfrac{t-3-t-h+3}{h(t-3)(t+h-3)}$ dist neg

$= \lim_{h\to 0} \dfrac{-\cancel{h}}{\cancel{h}(t-3)(t+h-3)}$ combine

$= \lim_{h\to 0} \dfrac{-1}{(t-3)(t+h-3)}$ cancel

$= \dfrac{-1}{(t-3)(t+0-3)}$ take limit by subst h=0!

$= \dfrac{-1}{(t-3)(t-3)}$

$g'(t) = \boxed{\dfrac{-1}{(t-3)^2}}$

b) $g'(2) = m_{TAN} = \dfrac{-1}{(2-3)^2} = \dfrac{-1}{(-1)^2} = \dfrac{-1}{1} = \boxed{-1}$

c) $y - y_1 = m(t - t_1)$ $y_1 = g(2) = \frac{1}{2-3} = \frac{1}{-1} = -1$

$y - (-1) = -1(t-2)$ $t_1 = 2$

$y + 1 = -t + 2$ $m = -1 = g'(2)$

$\boxed{y = -t + 1}$

or $y = mt + b$

$-1 = -1(2) + b$

$-1 = -2 + b$

$1 = b$

$\boxed{y = -t + 1}$

M250 2.1

Key points about the tricky algebra maneuers

- conjugate is the opposite sign between the radicals; signs inside radicals stay exactly the same

- FOIL numerator should cause middle terms to be the same, but opposite signs, and cancel out.

- Do not forget to write the conjugate in the denominator!

- Do not distribute $\left.\begin{matrix} h \\ (x-a) \\ \Delta x \end{matrix}\right\}$ in denominator -- you'll have to factor it out again to cancel it. $\frac{h}{h} = \frac{\Delta x}{\Delta x} = \frac{(x-a)}{(x-a)}$

- when multiplying $(-\sqrt{x-3})(+\sqrt{x-3})$ use parentheses $-(x-3)$ and distribute negative on next step. This prevents sign errors.

- EVERY term in the numerator, except $\left\{\begin{matrix} h \\ (x-a) \\ \Delta x \end{matrix}\right.$ terms, should disappear when combining like terms.

- Memorize the algebraic signposts of correct work
  - difference of squares removes radicals in numerator
  - all terms cancel after dist neg.
  - $\frac{\Delta x}{\Delta x}$ cancels out $\left(\text{or } \frac{h}{h} \text{ or } \frac{x-a}{x-a}\right)$.

- If you make an algebra error, one or more of these signposts won't happen.